

RELATION IDENTITIES EQUIVALENT TO CONGRUENCE MODULARITY

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ABSTRACT. We present some identities dealing with reflexive and admissible relations and which, through a variety, are equivalent to congruence modularity.

1. INTRODUCTION

Congruences and congruence identities have proved to be fundamental notions in universal algebra. See, e. g., Jónsson [CV] for an introduction. It has been observed that sometimes reflexive and admissible relations play an important role even when the main concern are congruences. See, e. g., [CV, p. 370] and Tschantz [T], just to mention some. Tolerances, too, have sometimes proved useful, see, for example, Chajda [Ch], Czédli, Horváth, and Lipparini [CHL], Kearnes and Kiss [KK], Lipparini [L] and further references there. Many identities equivalent to congruence modularity are known, e. g., the quoted [CHL, CV, T], Freese and Jónsson [FJ], Gumm [G1, G2] and further references in the quoted papers. We shall describe here some identities which are equivalent to congruence modularity but are expressed also in terms of reflexive and admissible relations. A sample of the identities we have found is given in the following theorem, but first we need to introduce some notations.

Juxtaposition denotes intersection, \circ denotes composition of binary relations and, for R a relation, R^\smile denotes the *converse* of R , that is, $b R^\smile a$ holds if and only if $a R b$. By R^* we denote the transitive closure of R and \overline{R} denotes the smallest reflexive and admissible relations containing R (of course, this is dependent on the algebra we are working in). Recall that a *tolerance* Θ is a reflexive, symmetric and admissible relation. For simplicity, at first reading, the reader might always take all tolerances here to be congruences.

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We say that a relation identity ε *holds* in some variety \mathcal{V} if, for every algebra $\mathbf{A} \in \mathcal{V}$, the identity ε holds for all reflexive and admissible relations of \mathbf{A} . Some variables in ε might be required to vary among tolerances or congruences; formally, this makes no difference, since if R is a variable for reflexive and admissible relations, then, say, $(R \circ R^\sim)^*$ can be considered as (or substituted for) a variable for congruences.

Notice that an inclusions such as $\iota \subseteq \iota'$ can be considered as an identity, since it is equivalent to $\iota = \iota'$.

Theorem 1.1. *For every variety, each of the following identities is equivalent to congruence modularity.*

- (1.1) $\Theta(S \circ S) \subseteq (\Theta S)^*$
- (1.2) $\Theta S^* \subseteq (\Theta S)^*$
- (1.3) $\Theta(S \circ S^\sim) \subseteq (\Theta S \circ \Theta S^\sim)^*$
- (1.4) $\Theta(S \circ T)^* \subseteq \Theta(\overline{S \cup T}) \circ (\Theta S \circ \Theta T)^*$
- (1.5) $\Theta(S \circ T) \subseteq \Theta(\overline{S^\sim \cup T}) \circ (\Theta S \circ \Theta T)^*$

where S and T vary among reflexive and admissible relations and Θ can be equivalently taken to vary among tolerances or congruences.

The result is quite curious since, by minimal variations on the above identities, we get identities which are not equivalent to congruence modularity. For example, if we “merge” (1.3) and (1.4) as $\Theta(S \circ T^\sim) \subseteq (\Theta S \circ \Theta T^\sim)^*$, we get an identity equivalent to congruence distributivity, hence strictly stronger than modularity. As another example, the variation $\Theta(S \circ S) \subseteq (\Theta S^\sim)^*$ of (1.1), too, is strictly stronger than modularity, since it implies m -permutability for some m .

The identities in Theorem 1.1 are special cases of the identities (A1), (B1), (C1) and (D1) in Corollary 3.1, which shall be proved below.

2. A STRONG IDENTITY FOR RELATIONS

H.-P. Gumm [G1, G2] provided a characterization of congruence modular varieties by means of the existence of certain terms; we shall not need the explicit description of Gumm terms in what follows. Kazda, Kozik, McKenzie and Moore [AdJt] showed that a variety has Gumm terms if and only if it has *directed Gumm terms*, that is, terms p, j_1, \dots, j_k satisfying the following set of identities, for some k .

- (DG1) $x = p(x, z, z)$
 (DG2) $p(x, x, z) = j_1(x, x, z)$
 (DG3) $x = j_i(x, y, x),$ for $1 \leq i \leq k,$
 (DG4) $j_i(x, z, z) = j_{i+1}(x, x, z)$ for $1 \leq i < k$
 (DG5) $j_k(x, y, z) = z$

In particular, by the mentioned results, a variety is congruence modular if and only if it has directed Gumm terms, for some k . Notice that we have given the definition of directed Jónsson terms in the reversed order, in comparison with [AdJt]. However, the two definitions are obviously equivalent: just simultaneously reverse both the order of variables and the order of terms.

Recall the notations introduced right before Theorem 1.1; in particular, recall that juxtaposition denotes intersection. Furthermore, we let $S \circ_m T$ denote $S \circ T \circ S \dots$ with m factors, that is, with $m - 1$ occurrences of \circ . Moreover, R^h is $R \circ R \circ R \dots$ with h factors, that is, $R^h = R \circ_h R$. We let $S + T$ denote $\bigcup_{m \in \mathbb{N}} S \circ_m T$; in particular, for α and β congruences, $\alpha + \beta$ is the *join* in the congruence lattice. Notice that the set of all reflexive and admissible relations on some algebra also forms a lattice, but in this case the join of S and T is $\overline{S \cup T}$. We shall frequently use the fact that $\overline{S \cup T} \subseteq S \circ T$, for reflexive and admissible relations S and T . Notice also that, in the above notations, for a reflexive relation R , we have $R^* = R + R$. If R is a reflexive and admissible relation, let Θ_R be the smallest tolerance containing R , that is, $\Theta = \overline{R \cup R^\sim}$.

Theorem 2.1. *If a variety \mathcal{V} has $k+1$ directed Gumm terms p, j_1, \dots, j_k , with $k \geq 2$, then, for every natural number $\ell \geq 1$, \mathcal{V} satisfies the following identities*

$$(1) \quad R(V \circ W)(S_1 \circ S_2 \circ \dots \circ S_\ell) \subseteq R(\overline{V \cup W}) \circ (\Theta_R S_1 \circ \Theta_R S_2 \circ \dots \circ \Theta_R S_\ell)^{2k-3}$$

$$(2) \quad R(V \circ W)(S_1 \circ S_2 \circ \dots \circ S_\ell) \subseteq RR^\sim(\overline{V^\sim \cup W}) \circ (\Theta_R S_1 \circ \Theta_R S_2 \circ \dots \circ \Theta_R S_\ell)^{k-1}$$

where $R, V, W, S_1 \dots$ vary among reflexive and admissible relations.

Proof. Suppose that \mathbf{A} is an algebra belonging to \mathcal{V} and that in \mathbf{A} we have $(a, c) \in R(V \circ W)(S_1 \circ S_2 \circ \dots \circ S_\ell)$, for certain reflexive and admissible relations R, V, \dots . Then $a R c$, $a V b W c$ and $a = a_0 S_1$

$a_1 S_2 a_2 \dots a_{\ell-1} S_\ell a_\ell = c$, for certain elements b, a_1, a_2, \dots . In order to prove (1), let us compute

$$\begin{aligned} a &= p(a, p(a\mathbf{a}b), p(aa\mathbf{b})) \overline{V \cup W} p(a, p(a\mathbf{b}b), p(aa\mathbf{c})) = p(a, a, p(aac)), \\ a &= p(a, a, a) = p(a, a, p(aa\mathbf{a})) R p(a, a, p(aa\mathbf{c})), \end{aligned}$$

where elements in bold are those moved by V , W or R and we have used (DG1). Moreover, $p(a, a, p(aac)) = j_1(a, a, j_1(aac))$, by (DG2), hence

$$(3) \quad a R(\overline{V \cup W}) j_1(a, a, j_1(aac))$$

For $h = 0, \dots, \ell - 1$, we have

$$j_1(a, \mathbf{a}_h, j_1(aa\mathbf{h}c)) S_h j_1(a, \mathbf{a}_{h+1}, j_1(aa\mathbf{h+1}c))$$

For sake of brevity, let $j^*(x, y, z) = j_1(x, y, j_1(xyz))$, thus j^* satisfies $x = j^*(x, y, x)$, by (DG3). Then

$$\begin{aligned} j_1(a, a_h, j_1(aa\mathbf{h}c)) &= j^*(a, a_h, c) = j^*(j^*(aa\mathbf{h+1}a), a_h, j^*(ca\mathbf{h+1}c)) \Theta_R \\ j^*(j^*(aa\mathbf{h+1}c), a_h, j^*(a\mathbf{a}h\mathbf{+1}c)) &= j^*(aa\mathbf{h+1}c) = j_1(a, a_{h+1}, j_1(aa\mathbf{h+1}c)) \end{aligned}$$

Hence $j_1(a, a_h, j_1(aa\mathbf{h}c)) \Theta_R S_h j_1(a, a_{h+1}, j_1(aa\mathbf{h+1}c))$, for $h = 0, \dots, \ell - 1$. Concatenating, and setting $\Lambda = \Theta_R S_1 \circ \Theta_R S_2 \circ \dots \circ \Theta_R S_\ell$ we get $j_1(a, a, j_1(aac)) \Lambda j_1(a, c, j_1(acc)) = j_1(a, c, j_2(aac))$, by (DG4).

By similar (and easier) arguments, we have $j_2(a, a, c) \Lambda j_2(a, c, c) = j_3(a, a, c)$, hence $j_1(a, c, j_2(aac)) \Lambda j_1(a, c, j_2(acc)) = j_1(a, c, j_3(aac))$. Iterating, $j_1(a, c, j_3(aac)) \Lambda j_1(a, c, j_4(aac)) \dots$. Concatenating again, we get

$$\begin{aligned} (4) \quad j_1(a, a, j_1(aac)) \Lambda^{k-1} j_1(a, c, j_{k-1}(acc)) &= \\ j_1(a, c, j_k(aac)) &= j_1(a, c, c) = j_2(a, a, c) \Lambda^{k-2} j_{k-1}(a, c, c) = c \end{aligned}$$

by (DG5). Putting together (3) and (4), we get $(a, c) \in R(\overline{V \cup W}) \circ \Lambda^{2k-3}$, thus equation (1) is proved.

The proof of equation (2) is much simpler. We have $a = p(a, b, b) \overline{V \cup W} p(a, a, c)$, $a = p(a, a, a) R p(a, a, c)$ and $a = p(a, c, c) R p(a, a, c) = j_1(a, a, c)$. Moreover, as above, $j_1(a, a, c) \Lambda^{k-1} j_{k-1}(a, c, c) = c$, hence (2) follows. \square

Notice that if $k = 1$ in the definition of directed Gumm terms, then p is a Maltsev term for congruence permutability. Since in a congruence permutable variety every reflexive and admissible relations is a congruence, all the considerations below will become trivial in case $k = 1$, so we can always suppose $k \geq 2$. Notice that the above arguments show that $\alpha \circ \beta = \alpha \cup \beta$ holds in a congruence permutable variety.

Corollary 2.2. *If a variety \mathcal{V} has $k+1$ directed Gumm terms p, j_1, \dots, j_k , with $k \geq 2$, then, for every natural number $h \geq 1$, \mathcal{V} satisfies the identities*

$$(5) \quad \Theta(S \circ_{2^h} S) \subseteq (\Theta S)^{q+1}$$

$$(6) \quad R(S \circ_{2^h} T) \subseteq R(\overline{S \cup T}) \circ (\Theta_R S \circ_q \Theta_R T)$$

$$(7) \quad \Theta(S \circ_{2^h} S^\sim) \subseteq \Theta S^\sim \circ_r \Theta S$$

where $q = (2^{h+1} - 2)(2k - 3)$, $r = 1 + (2^{h+1} - 2)(k - 1)$, R, S, T vary among reflexive and admissible relations and Θ varies among tolerances (or congruences).

Proof. The identity (5) is the particular case of (6) when $S = T$ and $R = \Theta$, hence we shall go directly to the proof of (6).

The case $h = 1$ of (6) follows from equation (1) in Theorem 2.1, taking $\ell = 2$, $V = S_1 = S$ and $W = S_2 = T$. Suppose now that (6) holds for some $h \geq 1$. Since 2^h is even, we have $S \circ_{2^{h+1}} T = (S \circ_{2^h} T) \circ (S \circ_{2^h} T)$. Taking $\ell = 2^{h+1}$, $V = W = S \circ_{2^h} T$, $S_1 = S_3 = \dots = S$ and $S_2 = S_4 = \dots = T$ in equation (1), we get $R(S \circ_{2^{h+1}} T) \subseteq R(S \circ_{2^h} T) \circ (\Theta_R S \circ_{2^{h+1}(2k-3)} \Theta_R T)$, since ℓ is even. By the inductive assumption, $R(S \circ_{2^h} T) \subseteq R(\overline{S \cup T}) \circ (\Theta_R S \circ_q \Theta_R T)$, hence, noticing that q is even, we get $R(S \circ_{2^{h+1}} T) \subseteq R(\overline{S \cup T}) \circ (\Theta_R S \circ_{q'} \Theta_R T)$, where $q' = q + 2^{h+1}(2k-3)$. But $q' = (2^{h+1} - 2)(2k - 3) + 2^{h+1}(2k - 3) = (2^{h+2} - 2)(2k - 3)$, what we had to show.

As for the last identity, in case $h = 1$, take $\ell = 2$, $R = \Theta$, $V = S_1 = S$, $W = S_2 = S^\sim$ in identity (2) in Theorem 2.1, getting $\Theta(S \circ S^\sim) \subseteq \Theta S^\sim \circ_{2k-1} \Theta S$. If the identity (7) holds for some $h \geq 1$, then, since $S \circ_{2^{h+1}} S^\sim = (S \circ_{2^h} S^\sim) \circ (S \circ_{2^h} S^\sim)$ (here we are using the fact that 2^h is even, for $h \geq 1$), we can apply equation (2) in Theorem 2.1 with $\ell = 2^{h+1}$, $R = \Theta$, $V = W = S \circ_{2^h} S^\sim$, $S_1 = S_3 = \dots = S$ and $S_2 = S_4 = \dots = S^\sim$ getting $\Theta(S \circ_{2^{h+1}} S^\sim) \subseteq \Theta(S \circ_{2^h} S^\sim) \circ (\Theta S \circ_{2^{h+1}(k-1)} \Theta S^\sim)$, since $(S \circ_{2^h} S^\sim)^\sim = S^{\sim\sim} \circ_{2^h} S^\sim = S \circ_{2^h} S^\sim$, using the fact that both $2^{h+1}(k-1)$ and 2^h are even. By the inductive hypothesis, $\Theta(S \circ_{2^h} S^\sim) \subseteq \Theta S^\sim \circ_r \Theta S$, hence we get $\Theta(S \circ_{2^{h+1}} S^\sim) \subseteq \Theta S^\sim \circ_{r'} \Theta S$, for $r' = r + 2^{h+1}(k-1)$ noticing that r is odd. But $r' = r + 2^{h+1}(k-1) = 1 + (2^{h+1} - 2)(k-1) + 2^{h+1}(k-1) = 1 + (2^{h+2} - 2)(k-1)$, what we had to show. \square

3. FURTHER EQUIVALENCES AND REMARKS

In order to provide a uniform notation for the results in the following corollary, let \circ_∞ be another notation for $+$. This is justified since $R \circ_\infty S = R + S = \bigcup_{n \in \mathbb{N}} R \circ_n S$. Recall that Θ_R denotes the smallest tolerance containing the relation R .

Corollary 3.1. *For a variety \mathcal{V} and every $m \geq 2$, possibly $m = \infty$, each of the following identities is equivalent to congruence modularity*

- (A1) $\Theta(S \circ_m S) \subseteq (\Theta S)^*$ *equivalently*, $(\Theta(S \circ_m S))^* = (\Theta S)^*$
- (A2) $\Theta(S \circ_m S) \subseteq \Theta S + \Theta S^\sim$
- (A3) $\Theta(S \circ_m S) \subseteq (\Theta(S^\sim \circ S))^*$
- (B1) $\Theta(S \circ_m S^\sim) \subseteq \Theta S + \Theta S^\sim$ *equiv.* $(\Theta(S \circ_m S^\sim))^* = \Theta S + \Theta S^\sim$
- (B2) $\Theta(S \circ_m S^\sim) \subseteq (\Theta(S^\sim \circ S))^*$ *equiv.* $(\Theta(S \circ_m S^\sim))^* = (\Theta(S^\sim \circ S))^*$
- (C1) $R(S \circ_m T) \subseteq R(\overline{S \cup T}) \circ (\Theta_R S + \Theta_R T)$
- (C2) $\Theta(S \circ_m T) \subseteq (\Theta(\overline{S \cup T}))^*$ *equiv.* $(\Theta(S \circ_m T))^* = (\Theta(\overline{S \cup T}))^*$
- (C3) $R(S \circ_m T) \subseteq R(T \circ \overline{S \cup T}) \circ (\Theta_R S + \Theta_R T)$
- (C4) $\Theta(S \circ_m T) \subseteq (\Theta(T \circ S))^*$ *equivalently*, $(\Theta(S \circ_m T))^* = (\Theta(T \circ S))^*$
- (D1) $R(S \circ_m T) \subseteq R(\overline{S^\sim \cup T}) \circ (\Theta_R S + \Theta_R T)$
- (D2) $R(S \circ_m T) \subseteq R(\overline{S \cup T})(\overline{S^\sim \cup T})(\overline{S \cup T^\sim})(\overline{S^\sim \cup T^\sim}) \circ (\Theta_R S + \Theta_R T)$
- (D3) $R(S \circ_m T) \subseteq R(\overline{S \cup S^\sim \cup T \cup T^\sim}) \circ (\Theta_R S + \Theta_R T + \Theta_R S^\sim + \Theta_R T^\sim)$
- (D4) $\Theta(S \circ_m T) \subseteq \Theta(T \circ S) + \Theta(T \circ T^\sim) + \Theta(S^\sim \circ S) + \Theta(S^\sim \circ T) + \Theta(S^\sim \circ T^\sim) + \Theta(T^\sim \circ S) + \Theta(T^\sim \circ T)$
- (D5) $\Theta(S \circ_m T) \subseteq \Theta((T + T^\sim) \circ S) + \Theta(S^\sim \circ S) + \Theta(S^\sim \circ (T + T^\sim))$

where S, T vary among reflexive and admissible relations, Θ can be equivalently taken to vary either among congruences or among tolerances and R can be equivalently taken to vary either among congruences or reflexive and admissible relations.

Proof. If one of the above conditions holds when Θ varies among tolerances, then it obviously holds when Θ varies among congruences. A similar observation applies to R . Moreover, in each line with two conditions, both conditions are obviously equivalent, since $*$ is a monotone and idempotent operator. In (B2), if $m = 2$, in order to get the right-hand identity, use the left-hand identity twice, both as it stands and with S^\sim in place S . A similar remark applies to (C4).

By considering congruences α, β and γ , taking $R = \Theta = \alpha$, $S = \beta \circ \alpha \gamma$ and $T = \beta$ in any one of the above identities, we get an identity of the form $\alpha(\beta \circ_n \alpha \gamma) \subseteq \alpha \beta + \alpha \gamma$, for some $n \geq 3$ (here it is fundamental to assume that $m \geq 2$). The most involved case is (D3): notice that both $\alpha \gamma \circ \beta \subseteq \alpha \gamma \circ \beta \circ \alpha \gamma$ and $\beta \circ \alpha \gamma \subseteq \alpha \gamma \circ \beta \circ \alpha \gamma$, hence $\overline{S \cup S^\sim \cup T \cup T^\sim} \subseteq \alpha \gamma \circ \beta \circ \alpha \gamma$, hence $\alpha(\overline{S \cup S^\sim \cup T \cup T^\sim}) \subseteq \alpha(\alpha \gamma \circ \beta \circ \alpha \gamma) = \alpha \gamma \circ \alpha \beta \circ \alpha \gamma$. Since, for $n \geq 3$, obviously $\alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha(\beta \circ_n \alpha \gamma)$, then from $\alpha(\beta \circ_n \alpha \gamma) \subseteq \alpha \beta + \alpha \gamma$ we get $\alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha \beta + \alpha \gamma$. Through a variety, this condition implies congruence modularity by Day [D].

Hence it remains to show that congruence modularity implies each of the identities in the corollary, in the stronger form in which Θ varies

among tolerances and R varies among reflexive and admissible relations. By the mentioned results from [G1, AdjT], we can assume that \mathcal{V} has directed Gumm terms, for some k . Then, for every finite m , Condition (C1) follows from equation (6) in Corollary 2.2. Of course, if (C1) holds for every finite m , then it holds also for $m = \infty$. All the conditions except (B1), (D1) and (D2) are consequences of (C1), by the obvious monotonicity properties of the operators present in the identities. (B1) is a consequence of equation (7) in Corollary 2.2.

In order to prove (D1), first notice that, by (C1), we have $R(S \circ_m T) \subseteq R(S \circ T) \circ (\Theta_R S + \Theta_R T)$. By taking $\ell = 2$, $V = S_1 = S$ and $W = S_2 = T$ in equation (2) in Theorem 2.1, we get $R(S \circ T) \subseteq \Theta(\overline{S^\sim \cup T^\sim}) \circ (\Theta S + \Theta T)$. Putting together the above identities we get (D1).

The proof of the stronger (D2) is slightly more involved. By (C1), we have $R(S \circ_m T) \subseteq R(\overline{S \cup T}) \circ (\Theta_R S + \Theta_R T) = R(\overline{S \cup T})(S \circ T) \circ (\Theta_R S + \Theta_R T)$. We now can take $\ell = 2$, $R(\overline{S \cup T})$ in place of R , $V = S_1 = S$ and $W = S_2 = T$ in equation (2) in Theorem 2.1, getting $R(\overline{S \cup T}) = R(\overline{S \cup T})(S \circ T) \subseteq R(\overline{S \cup T})(\overline{S^\sim \cup T^\sim})(\overline{S^\sim \cup T^\sim}) \circ (\Theta_R S + \Theta_R T)$, since $(\overline{S \cup T})^\sim = \overline{S^\sim \cup T^\sim}$. Moreover, since $S \cup T = T \cup S$, we can repeat the argument once again, getting (D2). \square

Remark 3.2. We have made an essential use of the results by Kazda, Kozik, McKenzie and Moore [AdjT] in order to prove Theorem 2.1, hence to prove equations (5), (6) in Corollary 2.2 and Condition (C1) in Corollary 3.1. However, the reader who knows (undirected) Gumm terms might easily see that the above arguments can be adapted to get proofs for equation (7) in 2.2 and for conditions (A2)-(B2) and (D3)-(D5) in Corollary 3.1 using just Gumm terms. This might be convenient when we want to evaluate the number of actual factors on the right-hand sides, since there might be varieties with a smaller number of Gumm terms rather than directed Gumm terms.

In a few cases, it is even enough to use just Day terms [D]. In fact, there is a relation identity which characterizes exactly the number of Day terms of a congruence modular variety. See the next proposition.

Recall that *Day terms* are quaternary terms d_0, d_1, \dots, d_k satisfying the following conditions.

$$\begin{aligned} x &= d_i(x, y, y, x) && \text{for every } i; \\ x &= d_0(x, y, z, w); \\ d_i(x, x, w, w) &= d_{i+1}(x, x, w, w), && \text{for } i \text{ even}; \\ d_i(x, y, y, w) &= d_{i+1}(x, y, y, w), && \text{for } i \text{ odd, and} \end{aligned}$$

$$d_k(x, y, z, w) = w.$$

Proposition 3.3. *A variety \mathcal{V} has $k + 1$ Day terms d_0, d_1, \dots, d_k if and only if \mathcal{V} satisfies the identity*

$$\Theta(S \circ S^\sim) \subseteq \Theta S \circ_{k-1} \Theta S^\sim$$

where S varies among reflexive and admissible relations and Θ can be equivalently taken to vary among tolerances or among congruences.

Proof. Suppose that \mathcal{V} has Day terms d_0, d_1, \dots, d_k and in some algebra in \mathcal{V} we have $(a, c) \in \Theta(S \circ S^\sim)$, for Θ a tolerance. Thus $a \Theta c$ and there is some b such that $a S b S^\sim c$, hence $c S b$. Then, say for k even, $a = d_1(a, a, c, c) S d_1(a, b, b, c) = d_2(a, b, b, c) S^\sim d_2(a, a, c, c) = d_3(a, a, c, c) S d_3(a, b, b, c) \dots d_{k-1}(a, b, b, c) = c$. Moreover, by an argument in Czédli and Horváth [CH],

$$\begin{aligned} d_i(a, a, c, c) &= d_i(d_j(abb\mathbf{a}), \mathbf{a}, c, d_j(cbb\mathbf{c})) \Theta \\ &\quad d_i(d_j(abb\mathbf{c}), \mathbf{c}, c, d_j(\mathbf{a}bb\mathbf{c})) = d_j(a, b, b, c), \end{aligned}$$

for every $i, j \leq k$. The above relations show that $(a, c) \in \Theta S \circ_{k-1} \Theta S^\sim$.

Conversely, suppose that α, β and γ are congruences and take $\Theta = \alpha$ and $S = \beta \circ \alpha \gamma$ in the identity in the statement of the proposition. Then $\alpha(\beta \circ \alpha \gamma \circ \beta) = \alpha(S \circ S^\sim) \subseteq \alpha S \circ_{k-1} \alpha S^\sim = (\alpha(\beta \circ \alpha \gamma)) \circ_{k-1} (\alpha(\alpha \gamma \circ \beta)) = (\alpha\beta \circ \alpha\gamma) \circ_{k-1} (\alpha\gamma \circ \alpha\beta) = \alpha\beta \circ_k \alpha\gamma$, since, both $\alpha\beta$ and $\alpha\gamma$ being congruences, $k - 2$ factors are absorbed in the last identity, hence we end up with exactly k factors. It is a standard fact implicit in [D] that, within a variety, the congruence identity $\alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha\beta \circ_k \alpha\gamma$ corresponds exactly to the existence of $k + 1$ Day terms. \square

For an appropriate value of r , the identity (7) in Corollary 2.2 can be obtained as a consequence of Proposition 3.3 and, according to the respective number of terms in some given variety, we might get a better bound. Conditions (A2)-(B2) in Corollary 3.1, too, can be obtained as a consequence of Proposition 3.3.

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This is a preliminary version, still to be expanded. It might contain inaccuracies (to be precise, it is more likely to contain inaccuracies than subsequent versions).

We have not yet performed a completely accurate search in order to check whether some of the results presented here are already known. Credits for already known results should go to the original discoverers.

Though the author has done his best efforts to compile the following list of references in the most accurate way, he acknowledges that the list might turn out to be incomplete

or partially inaccurate, possibly for reasons not depending on him. It is not intended that each work in the list has given equally significant contributions to the discipline. Henceforth the author disagrees with the use of the list (even in aggregate forms in combination with similar lists) in order to determine rankings or other indicators of, e. g., journals, individuals or institutions. In particular, the author considers that it is highly inappropriate, and strongly discourages, the use (even in partial, preliminary or auxiliary forms) of indicators extracted from the list in decisions about individuals (especially, job opportunities, career progressions etc.), attributions of funds, and selections or evaluations of research projects.

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